Long/Short Extensions: How Much is Enough?

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Abstract

Long/short ratios like 130/30 are an increasingly common way for the investment management industry to describe portfolios that are released from the long-only constraint. The ratio of a portfolio's long and short positions to net notional value is often the primary description of the strategy, replacing more traditional measures such as active risk. Unfortunately, managers and their clients may not understand the underlying parameters associated with the value of the long/short ratio, beyond the general recognition that the size of the extensions (e.g., 30 percent) and active risk are positively related. We develop a mathematical model to identify the factors that determine the size of the long/short extension, and illustrate the relationships using historical data on the S&P 500 benchmark, as well as current data on a variety of domestic and international equity benchmarks. The model confirms the basic intuition that the size of the long/short extension increases with the active risk target chosen by the manager, and decreases with the estimated costs of shorting. In addition, the model shows that the expected short extension for an unconstrained portfolio depends on average security risk, average pair-wise security correlation, the security weight concentration of the benchmark, the number of investable securities, and the assumed accuracy of security return forecasts. The model provides important perspectives on long/short strategies as the investment management industry continues to move away from more traditional long-only portfolios.

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Long/Short Extensions: How Much is Enough?

One of the major innovations in portfolio construction over the last decade has been the adoption of long/short strategies that allow managers to fully exploit the cross-sectional variation in forecasted security returns. Extensions of the Grinold and Kahn (1994) theory of active management by Clarke, de Silva, and Thorley (2002) and others have focused on the role of formal constraints in portfolio construction, particularly the negative impact of the long-only constraint. At the same time, innovations in prime brokerage practices and the acceptance of shorting by institutional fiduciaries has led to a proliferation of long/short strategies and products. Because they are new to many market participants, misconceptions about long/short strategies abound, some of which have been addressed in a recent paper by Jacobs and Levy (2007). The analytic model developed in this paper will further improve investors' conceptual understanding about the factors that determine the size of the short (and equivalent long) extension in long/short strategies.

The long/short extension model is based on the concept of the expected short weight for individual securities in the benchmark, similar to Sorensen, Hua, and Qian (2006). We also employ a constant correlation matrix assumption and other modeling techniques of an early analytic treatment of long/short strategies by Jacobs, Levy, and Starer (1998). This paper describes how the expected short weight for a security depends on the relative size of the security's benchmark weight and the active weight assigned to that security by the portfolio management process. The formal mathematical model and approximations enhance perspectives from previous studies that have depended on time-period specific numerical examples, or insights from simulations. The analytical model we derive shows that the size of the short extension is dependent on a number of security and benchmark parameters, in addition to the tracking error embedded in the portfolio strategy. An analysis of the impact of the various parameters of the model reveals several important concepts in long/short portfolio management. For example, three of the model parameters: individual security risk, security correlation, and benchmark weight concentration, change over time, suggesting that the exact value of the long/short ratio should be allowed to vary in order to maintain a constant level of active risk.

The derivation of the long/short extension model rests on the assumption of an unconstrained portfolio optimization, and thus gives an upper bound on possible long/short ratios in practice. In the language of the fundamental law of active management, we assume a transfer coefficient of one and thus the maximum possible expected information ratio. As discretionary constraints are imposed the long/short ratio declines from the upper bound suggested by the model, with a corresponding decline in information ratio. As a result, the empirical illustrations using the S&P 500 and other common equity benchmarks in this paper have long/short ratios that are generally higher than applied strategies, where a variety of additional constraints are often employed. Besides being difficult to model mathematically, the incorporation of formal constraints that vary from manager to manager would make our analysis less generic. Within the assumption of an unconstrained optimization process, we also discuss the special case of market-neutral or zero net-long portfolios, and use it to motivate a simple approximation of the general long/short model.

Our goal is to enhance past attempts to analyze the long/short ratio that have relied on Monte Carlo simulation or numerical optimization using representative data. Such studies by Sorensen, Hua, and Qian (2006) and Clarke, de Silva, and Sapra (2004) allow for consideration of a wider range of implementation issues, including discretionary constraints, but lack the generality of an analytic model. For example, numerical analyses of S&P 500 benchmarked long/short portfolios that are only a few years old may already be outdated due to shifts in key market parameters. We present the intuition for the derivation of the long/short extension model in the first section, with technical details covered in the appendix. The basic model assumes a simplified covariance matrix structure and an unconstrained optimization in the absence of costs. We then introduce a cost adjustment as well as two approximations of the basic model. In the second section we explore the impact of the various parameters in the model using the S&P 500 benchmark and associated security values from 1967 to 2006. We find, for example, that changes in security risk over time have had a greater impact on the size of the short extension than either benchmark concentration or security correlation. In the third section we apply the short extension model to current (i.e., year-end 2006) parameter values for various domestic and international equity benchmarks to explore how the number of securities and concentration impact the long/short ratio. Our summary and conclusions are in the last section.

The Short Extension Model

The analysis of the short extension in long/short portfolios is based on a decomposition of the security weights in the managed portfolio into benchmark and active weights. Specifically, the portfolio weight for the i^{th} security, w_{Pi}, can be defined as the sum of the security's benchmark weight, w_{Bi}, and active weight, w_{Ai},

$$w_{P\,i} = w_{B\,i} + w_{A\,i}\,. \tag{1}$$

While the benchmark weight for any given security is set by the market, the active weight is chosen by the manager. A basic tenet of portfolio theory is that the portfolio's expected active return (i.e., benchmark relative alpha) and active risk (i.e., tracking error) are a function of the *active* security weights, not the benchmark weights.

For optimized portfolios, the set of active security weights is determined by forecasted security returns, the estimated security return covariance matrix, and the targeted level of active portfolio risk. As noted in the Technical Appendix, a closed-form solution exists for optimal active weights in the absence of portfolio constraints. In this paper we assume a simplified covariance matrix where the N security risks are all equal to a single value, σ, and the $N\,(N\text{-}1)\,/\,2$ pair-wise correlations are all equal to a single value, ρ. As shown in the Technical Appendix, under this simplified covariance matrix the optimal active weights are a scalar multiple, c, of a set of standard normal scores, S_i,

$$w_{A\,i} = c\,S_i \quad \text{with} \quad c = \frac{\sigma_A}{\sigma\sqrt{1-\rho}\sqrt{N}} \tag{2}$$

where σ_A is the targeted level of active portfolio risk.

The zero-mean scores used in Equation 2 suggest that the active weight assigned to any given security can be thought of as a normal random variable, with a mean of zero and standard deviation of c. The multiplier c in Equation 2 shows that the range of active weights around zero increases with the portfolio's targeted active risk, σ_A, and decreases with N, the number of securities in the benchmark or investable set.[1] In addition, Equation 2 shows that under the simplified covariance matrix, the range of active weights decreases for higher security risk, σ, and increases with a higher correlation, ρ, between security returns. The dependence of the absolute magnitude of the active weights on these parameters will be instrumental to understanding the implications of parameter changes on the amount of shorting in the optimized portfolio.

Having described the active security weights, we turn our attention to the benchmark weights, $w_{Bi,}$ in Equation 1. By definition, the N benchmark weights are individually positive and sum to one. For standard capitalization-weighted benchmarks, the distribution of weights is also fairly concentrated; a few securities have large weights while many other securities have relatively small weights. When market-capitalization benchmark weights are sorted in descending order they generally decline in a geometric fashion, with the smallest benchmark weights approaching zero, as shown in Figure 1 for $N = 500$. We later formalize the assumption of a perfect geometric decline in benchmark weights, but for now simply focus on the intuition provided by sorting the benchmark weights from largest to smallest. Consider a single large benchmark-weight security towards the left-hand side of Figure 1, and the N possible active weights that might "randomly" be assigned to it by the manager's forecasting process. For a large benchmark-weight security, the probability that the assigned active weight is negative and large enough to lead to a negative total weight, or short position, is relatively low. On the other hand, for a small benchmark weight security towards the right-hand side of Figure 1, the probability of shorting is higher, approaching 1/2 for benchmark weights of zero. Similarly, the magnitude (as opposed to just the probability) of shorting depends on the relative magnitudes of the benchmark and active weights.

We now describe the concept of an "expected short weight" for each security based on a benchmark weight and a randomly assigned active weight. While much of the derivation that

[1] We generally refer to the N securities in the benchmark portfolio (e.g., $N = 500$ for the S&P 500 benchmark) but if the investor's universe is larger than the benchmark, and non-benchmark securities are given weights of zero, then N can refer to the number of securities in the investable set.

follows is relegated to the Technical Appendix, we include the initial steps in the body of the paper to emphasize how expected shorting depends on the relative magnitudes of the benchmark and active weights. Using probability theory, a security's expected short weight is the expected value of the total weight, w_{Pi}, conditional on it being negative, times the probability of being negative,

$$E(short_i) = E(w_{P\,i} \mid w_{P\,i} < 0) * \mathrm{Prob}(w_{P\,i} < 0)\,. \tag{3}$$

Using Equations 1 and 2, the probability of the total security weight being negative is the same as the probability that the z-score assigned to the security is negative enough to offset the benchmark weight divided by the scaling factor, *c*. In other words

$$E(short_i) = \left(w_{B\,i} + c \;\; E(S \mid S < -w_{B\,i} / c)\right) * \mathrm{Prob}(S < -w_{B\,i} / c)\,. \tag{4}$$

As described in the Technical Appendix, applying well-known standard normal probability functions gives the final result for a security's expected short weight as

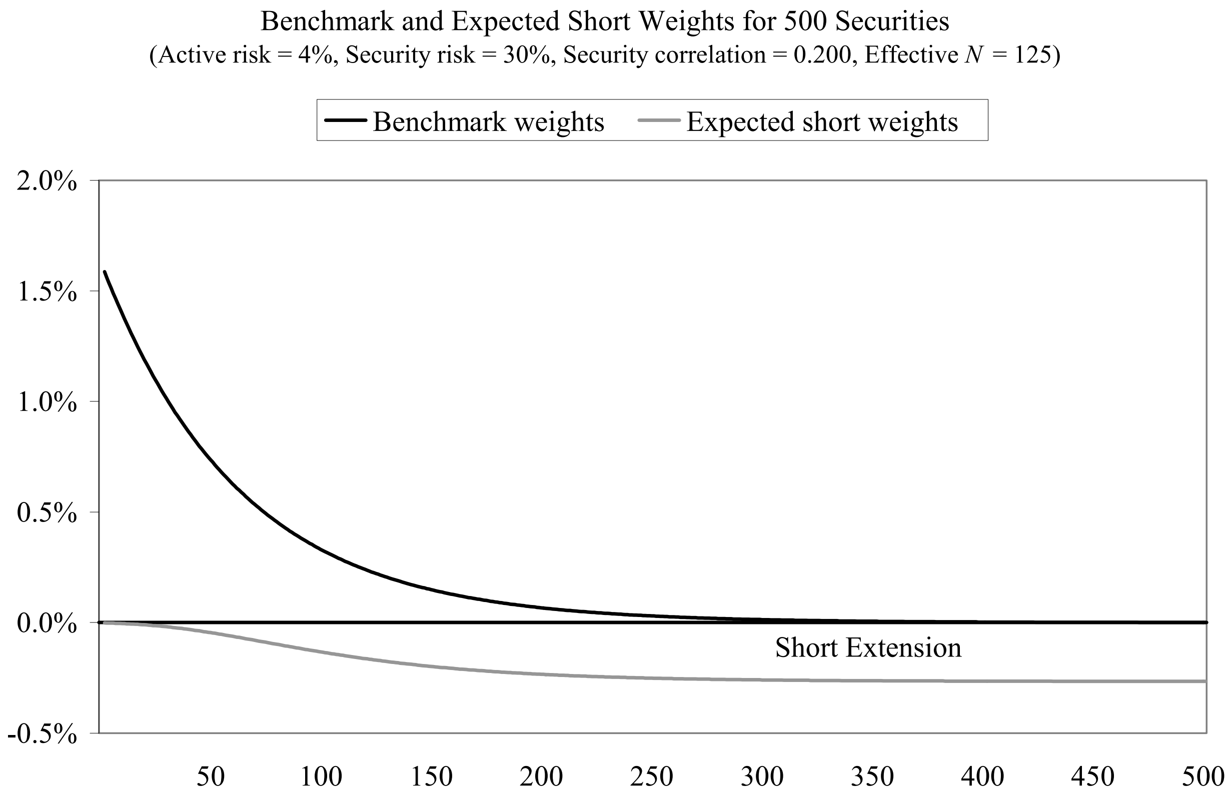
$$E(short_i) = c\ \varphi(-w_{B\,i} / c) - w_{B\,i}\ \Phi(-w_{B\,i} / c) \tag{5}$$

where Φ( ) is standard normal *cumulative* density function, and φ( ) is the standard normal density function. For ease of interpretation, the implicit short-sell induced negative sign in Equation 5 has been dropped so that larger positive values indicate more shorting. As illustrated in Figure 1, the expected short weight in Equation 5 is close to zero for large benchmark-weight securities, and asymptotically approaches a maximum value of

$$E(short_i \mid w_{Bi} = 0) \;=\; \frac{\sigma_A}{\sigma\sqrt{1-\rho}\sqrt{N}\sqrt{2\pi}} \tag{6}$$

for small benchmark-weight securities. Note that Equation 6 is not the largest possible short position; simply the average or *expected* short position for a zero benchmark weight security given all of the active weights that might be assigned to it by the manager's forecasting process.

Figure 1



The expected amount of shorting in the entire portfolio (e.g., the 30% implicit in a "130/30 portfolio") is the summation of Equation 5 across all N securities;

$$S_0 = \sum_{i=1}^{N} c\, \varphi(-w_{B\,i} / c) - w_{B\,i}\, \Phi(-w_{B\,i} / c)\,. \qquad (7)$$

Equation 7 is the basic model for the expected amount of unconstrained portfolio shorting in the absence of costs. The actual short extension in any given optimization will vary around the expected value in Equation 7 depending on how the active weights are assigned to the benchmark weights. While such assignments are certainly not random from the manager's perspective, Equation 7 is the average short extension across the very large number (N factorial) of possible assignments of N active weights to N securities. In other words, the size of the unconstrained short extension is a random variable, and we refer to the result in Equation 7 as the *expected* short extension for a given benchmark and active risk target.

Figure 1 provides a geometric interpretation of the portfolio's expected short extension in Equation 7 for an active risk of 4.0 percent, $N = 500$, and other parameter values indicative of the S&P 500 in recent years.[2] The horizontal axis in Figure 1 is the security rank when sorted in declining benchmark weight order. The vertical axis shows individual security weights, both the benchmark weights and expected short weights. The expected short extension for the portfolio is the area above the curve of expected short weights and below the horizontal axis, labeled "Short Extension." The long/short ratio depends on the size of this area compared to the area below the curve of benchmark weights and above the horizontal axis, which is exactly one (100 percent) by definition for any benchmark size or concentration. The geometry of Figure 1 can be used to illustrate how the long/short ratio will change with changes in the underlying parameter values. For example, if the benchmark becomes more concentrated in a few large securities, the curve of benchmark weights becomes steeper (while still enclosing an area of 100 percent) and the curve of expected short weight shifts to the left, increasing the area enclosed below the horizontal axis. Alternatively, an increase in the level of active risk does not affect the benchmark weights, but does increase the height of the short extension area, resulting in an increase in the long/short ratio.

The basic model of the short extension has at least one special case worth mentioning before we proceed with an analysis of costs. The focus of this paper is the short extension in fully invested portfolios, i.e., portfolios where the short weights are matched by long weights in excess of 100 percent of the portfolio's notional value. In market-neutral portfolios where the benchmark weight is zero by definition for all securities, Equation 7 becomes

$$\text{Market Neutral } S_0 = \frac{\sigma_A \sqrt{N}}{\sigma \sqrt{1-\rho}\sqrt{2\pi}} . \tag{8}$$

Equation 8 is simply the sum of Equation 6 across N securities, and provides a number of interesting perspectives on market-neutral portfolio construction. For example, the

[2] The benchmark weights used in Figure 1 are analytic weights, as discussed in the Technical Appendix, rather than actual S&P 500 weights. The analytic weights are constructed to have the same degree of security concentration as the S&P 500 benchmark at the end of 2006.

unconstrained amount of shorting in a market-neutral portfolio is linearly dependent on the active risk, σ_A, and the square root of the number of securities, $\sqrt{N}$. The form of the market-neutral special case in Equation 8 is similar to a recent model of portfolio leverage by Johnson, Kahn, and Petrich (2007). Indeed, a major conclusion of Johnson, Kahn, and Petrich (2007) is that the "gearing" or leverage of a market-neutral portfolio cannot be independently chosen, once an active risk target is specified, without reducing the transfer coefficient. Equation 7 indicates that this property also holds in the more general case of net-long portfolios.

Further analysis of the basic model of portfolio shorting in Equation 7 requires a parameter that measures the degree to which benchmark weights are concentrated in a few securities. In this paper we use Effective N, popularized by Strogin, Petsch, and Sharenow (1999), which can be thought of as the number of equal-weighted securities that would have the same diversification implications as the N actual benchmark weights. As explained in the Technical Appendix, Effective N is one over the sum of the benchmark weights squared, and ranges from $N_E = N$ for an equally-weighted benchmark, to $N_E = 1$ for a portfolio that is completely concentrated in one security. For example, at the end of 2006 the S&P 500 had $N_E = 125$ but reached a low of about $N_E = 80$ in 1999 when the U.S equity market was concentrated in technology stocks.

The expected shorting in a zero-benchmark weight security shown in Equation 6, and the concept of Effective N, motivates a simple approximation of the basic long/short model. Assume that the benchmark is distributed among N_E large capitalization securities that have no material potential for shorting, with zero weights on the other N - N_E securities. Under this "step function" assumption for the benchmark weights, the basic model in Equation 7 becomes

$$S_0 \approx \frac{\sigma_A \sqrt{N}\left(1 - N_E / N\right)}{\sigma\sqrt{1-\rho}\sqrt{2\pi}} \tag{9}$$

which is completely closed-form (i.e., no probability functions or summations). The geometric interpretation of Equation 9 in Figure 1 is a rectangle of length $N - N_E$ which approximates the irregularly shaped "Expected short weight" area. While simple and intuitive, Equation 9 is only a good approximation of the amount of shorting for low to moderate values of the N_E/N ratio. In the Technical Appendix, we derive a more robust approximation of the basic model in Equation 7 using the idea of an average benchmark security. The more robust approximation for the portfolio expected short extension is

$$S_0 \approx (N - N_A)\frac{c}{\sqrt{2\pi}} - \frac{1}{2}\left[1 - \frac{2}{N_E}\right]^{N_A} \tag{10}$$

where

$$N_A \approx 1 - \frac{N_E}{2}\ln\left(\frac{c}{\sqrt{2\pi}} N_E\right)$$

is the rank of the "average" security.

The basic model for the expected level of shorting in long/short portfolios has been derived without consideration for costs or discretionary constraints. Although the constraints applied to long/short portfolios are subject to managerial discretion, the costs associated with portfolio short extensions are dictated by market conditions, and can substantially reduce the optimal level of shorting. We next introduce a simple adjustment to the no-cost expected shorting model in Equation 7 that produces levels closer to those seen in practice. Costs, as well as constraints, distort the mathematics employed to model optimal active weights. Portfolio optimization problems under costs or constraints are mathematically intractable, and solutions generally require numerical optimization.

While optimal active weights under costs are difficult to model, we can determine the general level of expected shorting using a marginal benefit equals marginal cost argument from the objective function. Using this approach, the Technical Appendix shows that the expected short extension with costs is a function of the previously derived zero-cost short extension. Calculating S_0 using the basic model in Equation 7 (or one of the approximations in Equations 9 or 10) the expected short extension with costs is

$$E(S) = S_0\left(1 - \frac{B + 2T}{IC\sqrt{N}\,\sigma_A}\right) \tag{11}$$

where *IC* is the manager's information coefficient, and *B* and *T* are cost parameters defined below. The information coefficient is the expected cross-sectional correlation between forecasted and realized security alphas, and is a commonly used measure of forecasting accuracy. The *IC* becomes relevant under costs because higher confidence in security return forecasts entices the investor to incur costs to achieve a greater expected active return, as described in the fundamental law of active management (Grinold and Kahn, 1994). Indeed, the denominator of the second term in Equation 11 is simply the expected active portfolio return before costs, $E(R_A)$, as specified in the fundamental law equation.

Cost as a percent of the dollar amount of shorting comes in two forms. First, *B* is the borrowing cost, or "haircut" difference between the interest rate paid to leverage long positions and the rate earned on short-sell proceeds. Borrowing costs vary with the difficulty the prime broker has in finding shares to lend out, but for S&P 500 securities *B* can be roughly approximated as 50 basis points. In addition to the explicit borrowing costs, portfolio short extensions together with the counter-balancing long-extension drive up the general cost of managing a portfolio. We make the simplifying but reasonable assumption that general operating costs increase linearly with leverage. For example, for any given level of transaction costs and turnover, a 130/30 strategy has approximately 160 percent of the operating cost of an equivalent long-only strategy. We use the notation *T* for the operating cost for an equivalent long-only strategy, which can vary widely depending on turnover, transaction costs, and other operational considerations. For S&P 500 benchmarked portfolios with 100 percent turnover per year and 40 basis points of round-trip transaction costs, the value of *T* would be about 40 basis points. The *T* in Equation 11 is multiplied by 2 because the incremental costs associated with the short extension must be counter-balanced with an equivalent long extension. Thus, increases in the level of shorting have a total *incremental* costs of 50 + 2(40) = 130 basis points. Note that we only use 130 basis points as a rough estimate of costs in the numerical examples of this study; our objective is not to precisely estimate the costs of shorting, but to model how costs reduce the expected short extension.

The information coefficient, IC, measures a portfolio manger's self-assessed accuracy in forecasting security returns, and as such is easily overstated, as explained in Grinold and Kahn (1994). In practice, IC is used by quantitative managers to properly scale the security alphas supplied to a numerical optimizer. Appropriate IC values for modeling purposes should be calibrated in conjunction with the number of securities using the fundamental law relationship $IR = IC\sqrt{N}$, where IR is the information ratio, defined as expected active return, $E(R_A)$, over active risk, σ_A. Grinold and Kahn (1994) argue that an $IR = 0.50$ is "good", and designate an $IR = 0.75$ as "very good" and an $IR = 1.00$ as "exceptional." This framework is also adopted in Goodwin's (1998) review of uses and interpretations of the information ratio. For the base case, we use an active risk of $\sigma_A = 4.0$ percent and choose $IR = 0.75$, so that $IC = 0.034$ when $N = 500$. Under the cost assumption of $B + 2T = 1.3\%$, Equation 11 indicates that expected short extension with costs is $E(S) = S_0\,(1 - .013/(0.75*0.04)) = S_0\,(0.57)$, or about 57% of the zero-cost model. We note that tests using a commercial optimizer with cost functionality generally confirm Equation 11 for S&P 500 portfolios.

In practice, managers apply a wide variety of discretionary portfolio constraints which may impact the level of shorting. For example, managers may explicitly constrain shorting with the understanding that moderate restrictions have only a minor impact on the expected active portfolio return, as measured by the transfer coefficient (see Clarke, de Silva, and Sapra (2004)). Other common constraints, such as limits to individual active weights or style and sector neutrality constraints may indirectly reduce portfolio shorting. We do not consider constraints beyond the requisite budget and active risk restrictions, in order to keep the analysis as generic as possible. Because we model unconstrained portfolios, the expected short extension numbers in this paper are higher than many applied strategies where additional constraints are employed.

Model Parameters and Implications: Illustration using Historical S&P 500 Data

In this section we explore the impact of the various parameters identified in the short extension model, including the cost adjustment in Equation 11. The position of the portfolio active risk parameter, σ_A, in the numerator of the simple approximation in Equation 9 verifies the common intuition that the size of the short extension increases with the manager's target for active risk (i.e., benchmark tracking error) in the optimization process. A higher active risk target translates into larger absolute magnitudes for the active security weights, w_{Ai}, as shown in Equation 2. The larger negative active weights naturally lead to more shorting, with a commensurate increase in long security weights to keep the portfolio in a 100 percent net-long balance. The positive impact of active risk on the level of shorting is reinforced when costs are considered, as shown by the position of the σ_A in the denominator of the last term in Equation 11. The active risk parameter is unique in the long/short extension model as the only true choice variable selected by the manager. Most other model parameters (e.g., security risk, benchmark concentration, costs) are exogenous in that they are "forced" on the manager by the market or choice of benchmark.

Figure 2

Short Extension and Active Risk for S&P 500 Benchmarked Portfolios

(Security Risk = 30%, Security Correlation = 0.200, Effective N = 125, Cost = 1.3%)

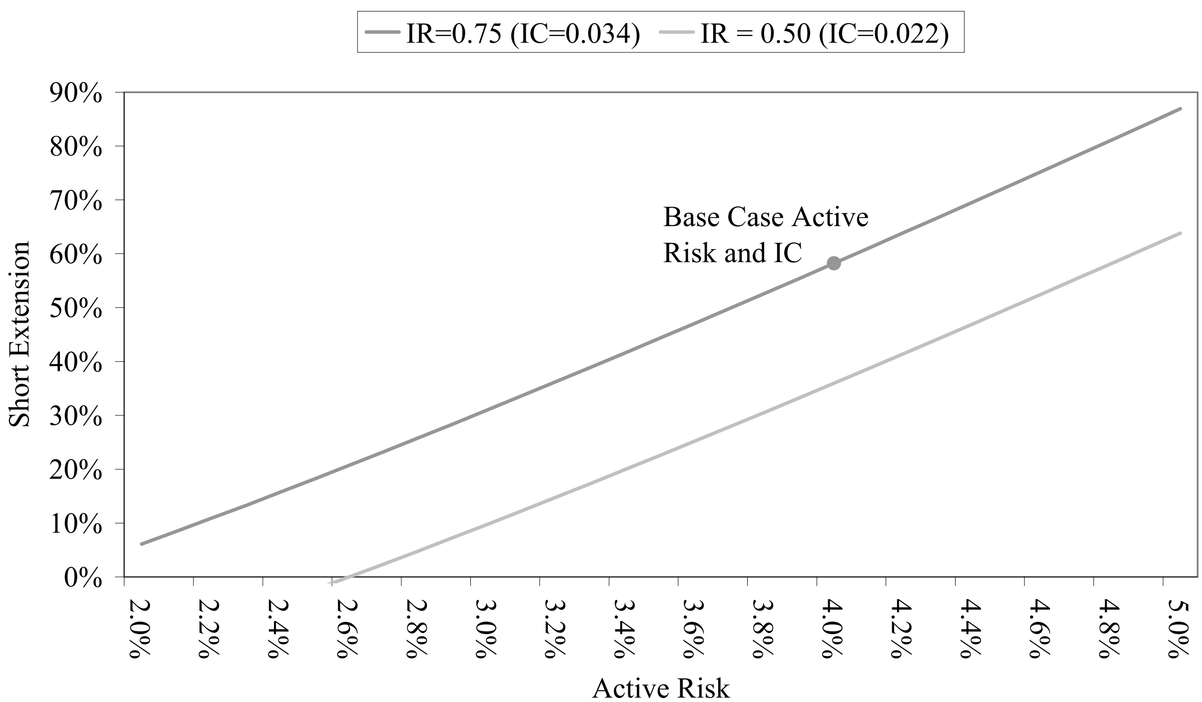


Figure 2 plots the short extension as a function of active portfolio risk for the basic model with a cost adjustment using the example S&P 500 parameter values from Figure 1 and a cost of

1.3 percent.[3] For this and the other illustrations that follow we use the calculation in Equation 7 instead of an approximation, although the differences are not generally visible. At the base-case ex-ante information ratio of 0.75 (and associated *IC* of 0.034), the expected short extension at 3 percent active risk is about 30 percent (a 130/30 portfolio), but over 50 percent at the base-case active risk of 4 percent. While there is some slight curvature, the relationship between expected short extension and active risk in Figure 2 is nearly linear, in accordance with the simple approximation in Equation 9.

Figure 2 also plots the short extension for various active risk levels using a lower ex-ante information ratio (*IR* = 0.50) to illustrate the impact of the *IC* parameter, which measures the manager's return forecasting accuracy. According to the fundamental law, a lower *IC* leads to a lower expected active return, before costs. A lower expected active return decreases the marginal benefit of the unconstrained optimal active security positions, and thus lowers the level of shorting after costs as shown in Equation 11 and illustrated in Figure 2. We note again that the model is for unconstrained portfolio optimizations, and may produce expected short extensions that are higher than those for long/short strategies under the variety of additional discretionary constraints used in practice.

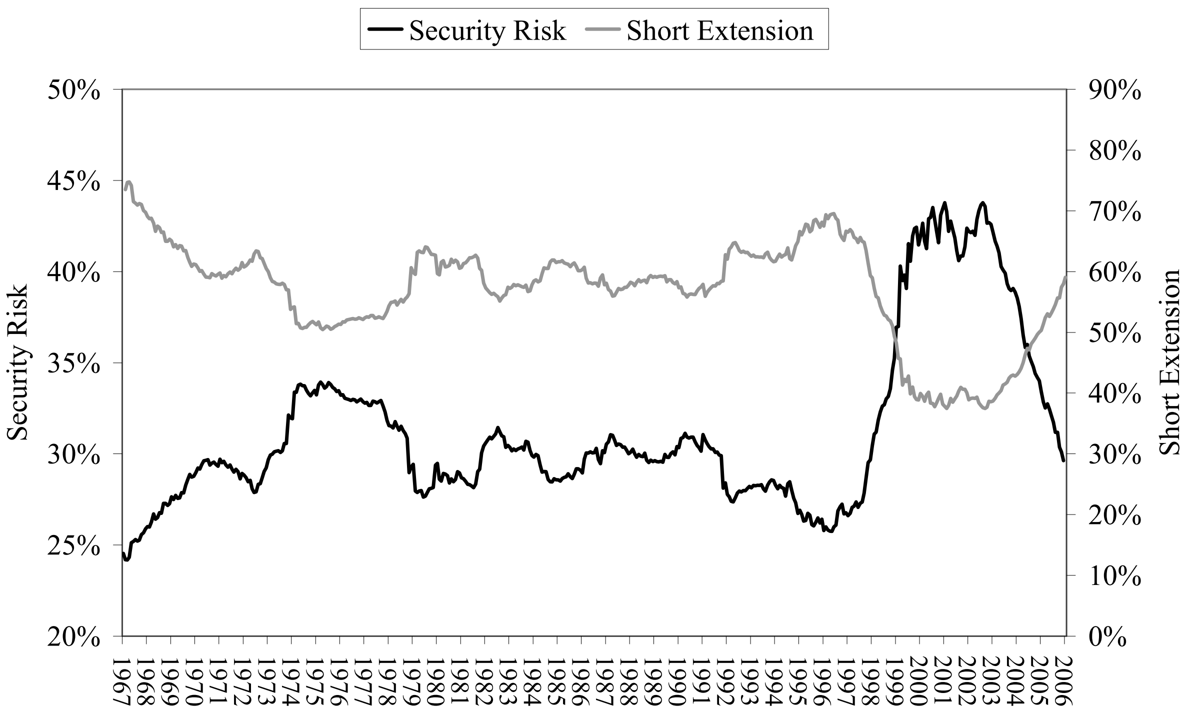
We next employ the model to analyze how the unconstrained short extension for S&P 500 benchmarked portfolios changes with three market parameters; security risk, security correlation, and benchmark concentration. Figures 3, 4, and 5 show the impact of historical changes in each of these three parameters holding the other two fixed. For example, the dark line in Figure 3 plots the average risk of S&P 500 securities each month from the end of 1967 to the end of 2006. Security risk is estimated by a trailing 60-month sample covariance matrix for the largest 500 common stocks in the CRSP database. The starting date of 1967 is determined by the first full year of market coverage (i.e., both NYSE and AMEX) of 1962 in the CRSP database, and the 60-month prior return requirement. Besides being fully populated (i.e., not just two parameters) estimated covariance matrices used in practice are typically based on more

[3] As in Figure 1, we use analytic benchmark weights that decline geometrically at a rate consistent with the S&P 500 Effective *N*, as opposed to actual security weights. In addition, our empirical analysis is based on the largest 500 U.S. common stocks from the CRSP database at each point in time, rather than exact S&P 500 membership.

sophisticated multi-factor risk models and/or GARCH time series analysis. Thus, the historical plots in Figures 3 may overstate the variation in estimated security risk derived from a more sophisticated model.

Figure 3

Security Risk and Short Extension: S&P 500 with 4% Active Risk
(Security Correlation = 0.200, Effective N =125, Cost = 1.3%, IC = 0.034)



The average security risk in Figure 3 varies over time between 25 percent and 35 percent, with the exception of the notable rise and fall associated with the build up and bursting of the technology bubble in the late 1990s. The lighter line in Figure 3 plots the expected short extension from the general model in Equation 7 with the cost adjustment in Equation 11 using the default parameter values and historical levels of security risk. As shown by the position of the security risk parameter, σ, in Equation 9, the long/short extension *decreases* with an increase in estimated security risk. Higher security risk decreases the absolute magnitude of the optimal active weights, leading to smaller short positions and a commensurate decrease in long weights.[4]

[4] Note that an increase in security risk, σ, increases the magnitude of security alphas under the Grinold (1994) prescription, but the optimization process that translates security alphas into optimal active weights effectively divides by σ^2 so that the net effect is a decrease in the size of active weights.

For example, the dramatic increase in security risk beginning in 1998 leads to a drop in the unconstrained expected short extension from about 70 percent to 40 percent (i.e., a 140/40 portfolio) by the year 2000, followed by a rise in expected shorting as individual security risk reverts back to long-term norms.

Figure 4

Security Correlation and Short Extension: S&P 500 with 4% Active Risk
(Security Risk = 30%, Effective N =125, Cost = 1.3%, IC = 0.034)

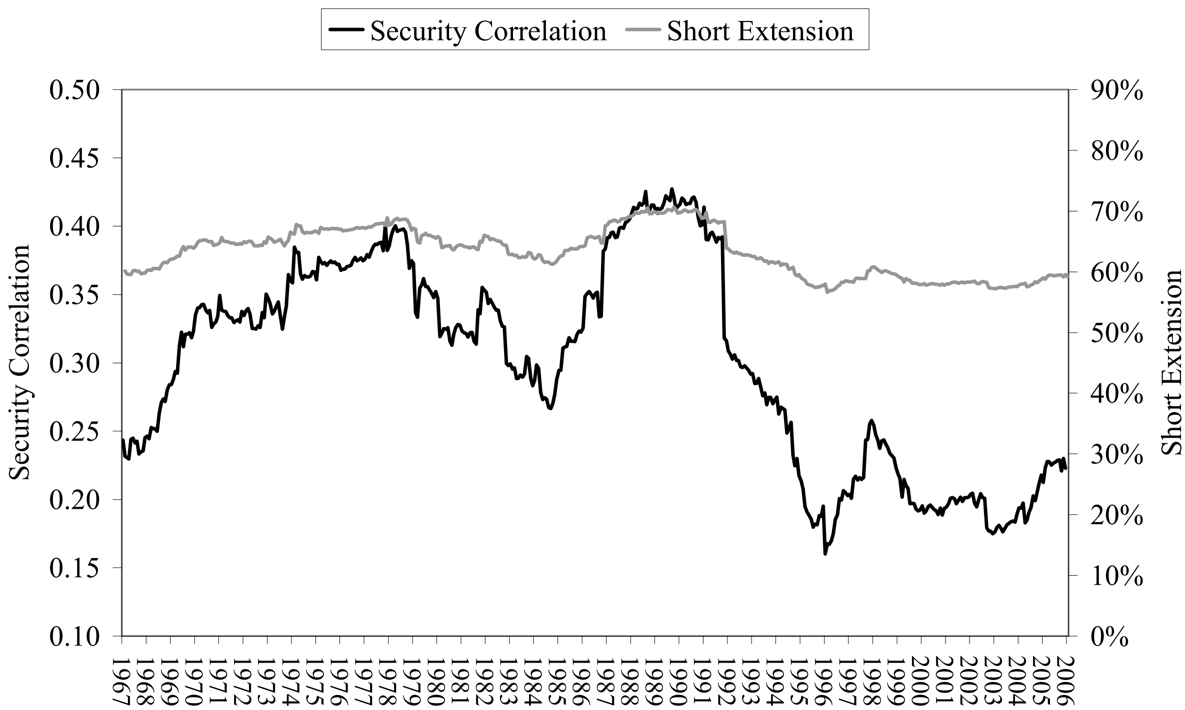


Figure 4 considers the impact of historical changes in the security correlation parameter, ρ, on the level of shorting, holding other parameters fixed. The dark line in Figure 4 shows the average pair-wise security correlation from 1967 to 2006 using the covariance matrix calculated from the trailing 60 months. While average security correlations at the end of 2006 are about 0.20, the levels have historically been higher, with a notable jump and then drop five years later associated with the inclusion and exclusion of the October 1987 observation. The lighter line in Figure 4 plots the expected short extension using the default parameter values and historical levels of security correlation. The short extension *increases* with an increase in security correlation as shown by the position of the ρ parameter in Equation 9. Higher correlations between securities translate into lower "breadth" in the Grinold and Kahn (1994) sense, so that larger active weights are required to maintain the targeted level of active risk, as shown in Equation 2. For example, the higher security correlations in the five-year post-crash period are associated with higher levels of shorting in Figure 4. The short extension in Figure 4 moves between about 60 and 70 percent over the historical range of security correlation values, a much smaller range than in Figure 3 for changes in security risk.

Figure 5

Effective N and Short Extension: S&P 500 with 4% Active Risk
(Security Risk = 30%, Security Correlation = 0.200, Cost = 1.3%, IC = 0.034)

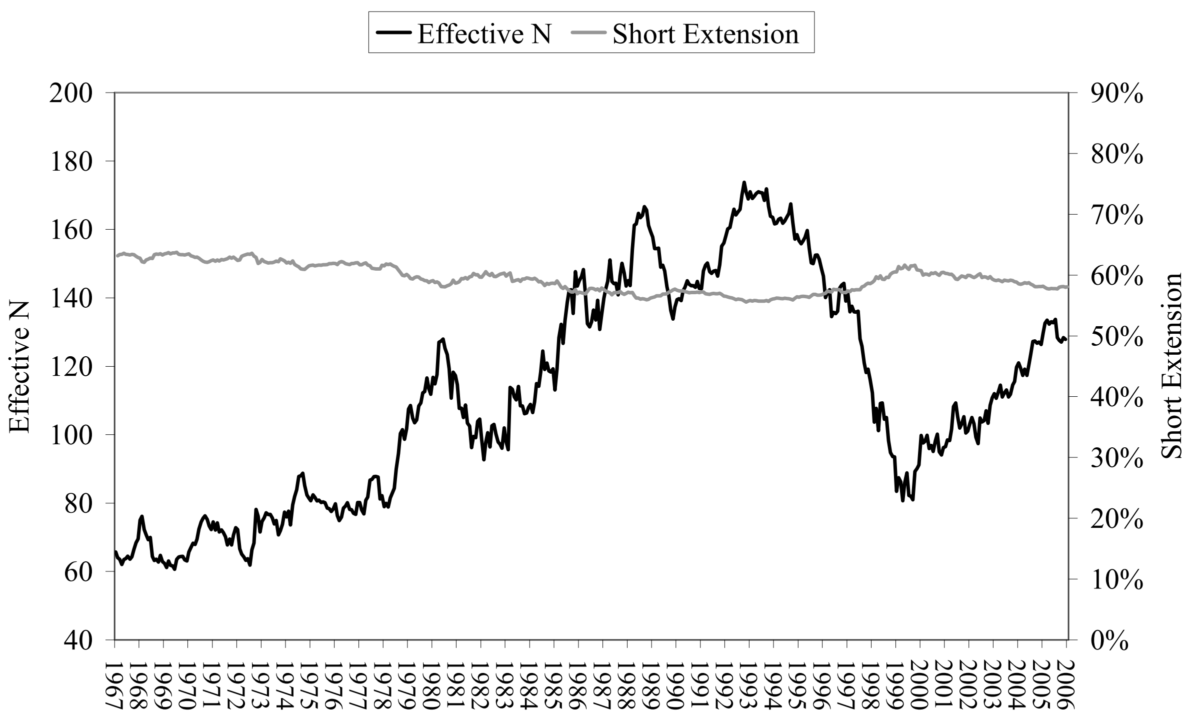


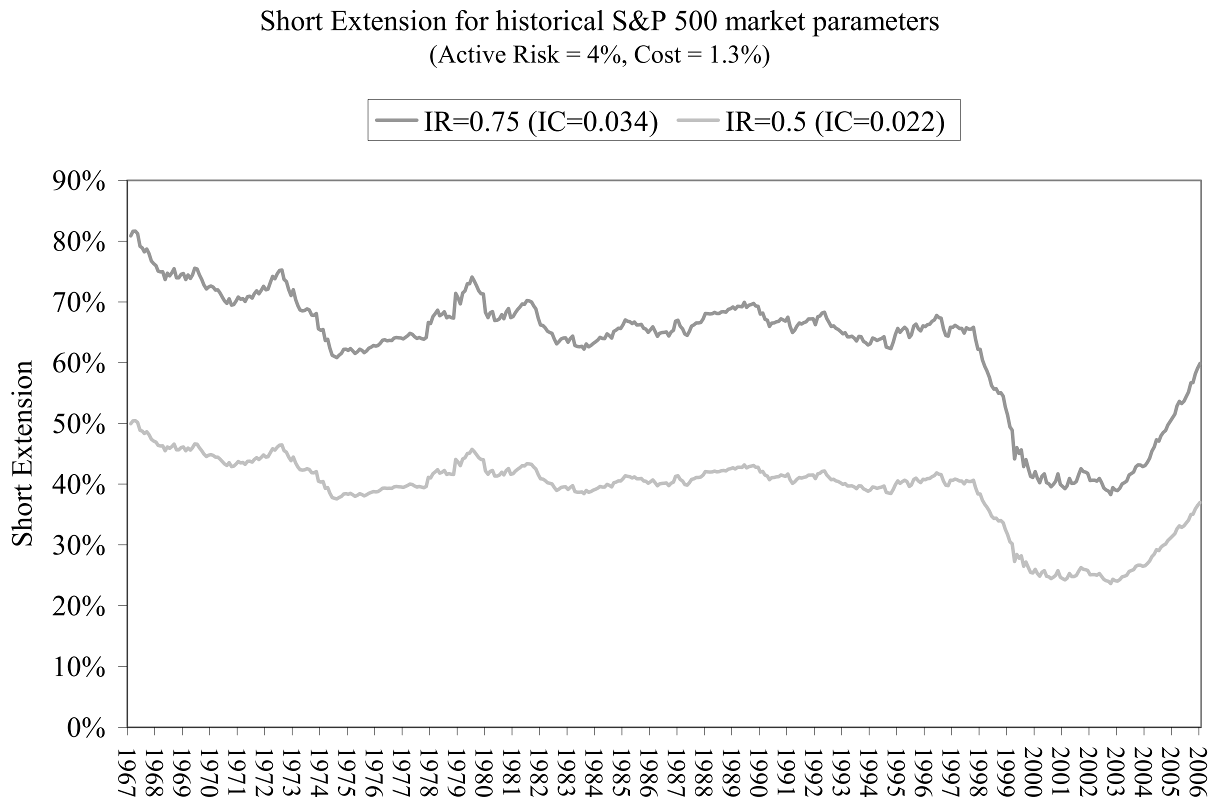
Figure 5 considers the impact of historical changes in the benchmark concentration holding the other parameters fixed. The dark line in Figure 5 plots monthly observations of Effective N for the S&P 500 benchmark from 1967 to 2006. While the historical risk illustrations in Figures 3 and 4 may overstate the actual variation of a more sophisticated ex-ante risk model, the Effective N values shown in Figure 5 capture the actual concentration of the S&P 500 benchmark at each point in time. Figure 5 shows a generally decreasing level of concentration (increasing Effective N) in the 500 largest U.S. stocks over time, going from an

Effective N of a little over 60 in the late 1960s to about 170 by 1993. The trend towards less concentration was reversed during the 1990s with the Effective N dropping to about 80 in the late 1990s as the U.S. equity market become dominated by a relatively few large-cap technology stocks. With the bursting of the technology bubble at the turn of the century, the market has again moved to less concentration with an Effective N of about 125 in 2006.

While the role of active risk in long/short extensions is perhaps the best understood of the model parameters, the impact of benchmark concentration on the size of long/short extension may be one of the least understood effects captured by the model. As more securities approach zero benchmark weights, they are prone to more shorting. For example, in a hypothetical set of 500 securities where 100 have benchmark weights of 1 percent, and the rest zero, only 400 securities are subject to meaningful shorting. On the other hand, if all the benchmark weight is shifted to 10 securities with weights of 10 percent, then 490 securities are effectively subject to shorting. Note that Effective N enters Equation 9 as a ratio to the actual number of securities, N_E/N, and that expected shorting is proportional to 1 - N_E/N. Despite fairly dramatic changes in security concentration, the expected level of shorting in Figure 5, holding other parameters fixed, moved in a relatively narrow range around 60 percent. The short extension increases with an increase in benchmark concentration (i.e., decrease in Effective N) although the variation based on historical values is small compared to security risk.

Figure 6 shows the *simultaneous* impact of the three market parameters over time on the expected level of shorting for an S&P 500 benchmarked portfolio at 4 percent active risk. The variation in the short extension is dominated by the changes in security risk from Figure 3, but the combined effect of all three parameters shows greater variation in shorting than any single parameter alone. Within recent history (i.e., 2003 to 2006) the short extension at the base-case active risk of 4 percent and IC of 0.034 (information ratio of 0.75) increased from about 40 percent to 60 percent. Figure 6 also plots the historical short extension at the lower IC value of 0.022 to re-emphasize the role of the manager's self-assessed forecasting skill. As shown in Figure 6, the level of shorting is smaller under the lower IC. In addition, the recent three year (2003 to 2006) range in short extensions is narrowed to between 25 and 40 percent (i.e., between 125/25 and 140/40 portfolios.)

Figure 6



The substantial changes in the general level of shorting based on a manager-specific parameter like information coefficient illustrates that the value of the analytic model is in identifying relevant parameters, direction of impact, and observed variation in value, not necessarily specifying the exact level. Discretionary constraints, a lower assumed information coefficient, and higher costs, all reduce the desirable level of shorting for a given strategy. One major implication of our analysis is that because relevant market parameters change over time, managers should allow the short extension to vary with market conditions even though the targeted level of active risk is held constant.

Model Parameters and Implications: Illustration using Other Benchmarks

In the prior examples we focused on long/short strategies based on the S&P 500 benchmark to illustrate the impact of the security risk, security correlation, and benchmark concentration parameters on the size of the long/short extension. The analytic model is equally applicable to other equity benchmarks, and Table 1 provides summary data on several domestic and international benchmarks at the end of 2006.[5] The comparison of long/short extension levels for different benchmarks illustrates the impact of *N* (number of investable securities) in the model, and includes more variation in benchmark concentration than was observed in the historical S&P 500 data.

Table 1

Equity Benchmarks and Market Parameters: December 2006

Benchmark	Number of Securities	Effective N	N_E/N Ratio	Security Risk	Security Correlation
S&P 500	500	129	0.257	31.1%	0.249
Nikkei 225	225	88	0.329	33.9%	0.363
FTSE 100	101	35	0.349	28.6%	0.277
MSCI EAFE	1164	247	0.212	32.5%	0.207
MSCI Japan	382	83	0.217	33.7%	0.330
MSCI Europe	601	144	0.240	32.0%	0.231
Russell 1000	987	157	0.159	32.9%	0.225
Russell 1000 Value	611	73	0.119	30.6%	0.241
Russell 1000 Growth	683	130	0.191	35.2%	0.230
Russell 2000	1972	1255	0.636	44.4%	0.186

In addition to the S&P 500 in the U.S., we include two other popular country market indices; the Nikkei 225 in Japan and the FTSE 100 in the United Kingdom. The Nikkei and FTSE are both smaller than the S&P 500 in terms of the number of securities in the benchmark, but have similar security concentrations as measured by the ratio of Effective *N* to *N*. We next include three MSCI regional indices, EAFE, Japan, and Europe, with a large range of *N* values but again, similar benchmark concentrations and security risk parameters. The last section of Table 1 includes four Russell indices for the U.S. domestic equity market; the large-cap Russell 1000, the Russell 1000 Value and Growth benchmarks, and the small-cap Russell 2000. The

[5] The data in Table 1 is from the Barra Morgan Stanley Global Equity Model (GEM) on December 29, 2006. The security correlation number was inferred from the average security risk value and the Barra estimated risk of an equally-weighted benchmark portfolio.

Russell 2000 is our only benchmark example that specifically excludes larger-capitalization securities, and consequently has a remarkably different concentration profile. While the Russell 2000 is a capitalization-weighted benchmark, the ratio of Effective N to N (1255/1972) at 0.636 is much higher than the other benchmarks, approaching the concentration profile of an equally-weighted benchmark.

Table 2

Equity Benchmarks and Expected Short Extension

Benchmark	Active Risk = 3% General Model (Equation 7)	Active Risk = 4% General Model (Equation 7)	Active Risk = 4% Simple Approximation (Equation 9)	Active Risk = 4% Robust Approximation (Equation 10)
S&P 500	31%	58%	56%	55%
Nikkei 225	17%	33%	30%	30%
FTSE 100	12%	23%	24%	22%
MSCI EAFE	47%	88%	84%	85%
MSCI Japan	26%	49%	50%	47%
MSCI Europe	33%	62%	60%	59%
Russell 1000	45%	82%	82%	80%
Russell 1000 Value	39%	72%	74%	70%
Russell 1000 Growth	33%	61%	62%	59%
Russell 2000	39%	76%	36%	72%

Table 2 provides calculations of the expected short extension using the parameter values in Table 1. For purposes of direct comparison, we use the S&P 500-based cost estimate of 1.3% for all benchmarks, although the borrowing and transaction costs will likely vary for different benchmarks. We also use a constant information ratio of 0.75 across benchmarks which leads to different information coefficients for each benchmark according to the fundamental law relationship $IR = IC\sqrt{N}$. The first column in Table 2 shows the expected short extension for each benchmark at a relatively low active risk target of 3 percent, using the general model in Equation 7 with the cost adjustment in Equation 11. For example, the expected short extension for an S&P 500 benchmarked portfolio at 3 percent active risk is 31 percent (a long-short ratio of

131/31). The expected short extension for the other benchmarks range from a low of 12 percent for the FTSE (N=100) to highs of 47 percent for the EAFE (N=1164) and 45 percent for the Russell 1000 (N=987) benchmarks. The comparison of the results between benchmarks shows that the size of the expected short extensions increases with the number of securities in the benchmark. Specifically, holding fixed the information ratio, $IR = IC\sqrt{N}$, in the cost adjustment in Equation 11, and the ratio N_E/N in Equation 9, the remaining $\sqrt{N}$ term in Equation 9 is a result of two effects. First, an increase in N decreases the size of the average benchmark weight (i.e., $1/N$) which increases the likelihood that the average security is shorted. Second, as the size of the investable set increases, active weights decline by $\sqrt{N}$ as shown in Equation 2. The net effect is an increase in expected shorting.

Table 2 next calculates the expected short extension at the base-case 4 percent active risk level, using the general model in Equation 7 and the two approximations in Equations 9 and 10, all with the cost adjustment in Equation 11. By comparing the second and third columns of Table 2, we find that the simple approximation in Equation 9 provides a reasonably accurate estimate of the general model for most of the benchmarks (e.g., 56 percent compared to 58 percent for the S&P 500). The one instance of poor accuracy for the simple approximation is the Russell 2000, with a value of 36 percent from Equation 9 compared to 76 percent for the general model in Equation 7. As mentioned previously, the simple approximation in Equation 9 is not robust to the entire range of possible benchmark concentrations, and we recommend using the more robust approximation in Equation 10 which gives an estimate of 72 percent, closer to the 76 percent result for the general model.

Summary and Conclusions

We have developed a mathematical model to analyze the parameters that impact the long/short ratio in unconstrained portfolios, and illustrated the relationships using historical examples of S&P 500 benchmarked portfolios as well as current examples for a variety of other equity benchmarks. The analytic model of the short extension is based on simplifying assumptions about the structure of the security covariance matrix used to optimize the active portfolio, as well as the concentration profile of the benchmark. Under these simplifying assumptions we are able to derive equations that specify the expected size of the short extension for long/short portfolios in the absence of constraints. While practitioners use live data, numerical optimizers, and a variety of constraints in actual application, the analytic model allows for a better understanding of the factors that impact the size of the short extension. For example, the mathematical model can provide important insights as investors adopt long/short strategies to a wider variety of equity benchmarks.

The mathematical model captures two parameters which are intuitively important in determining the size of the short extension: the short extension increases with the active risk of the strategy and decreases with the cost of shorting. In addition, the model identifies the role of three market parameters that change over time: security risk, security correlation, and the concentration of the benchmark as measured by Effective *N*. The unconstrained expected short extension decreases with security risk, increases with security correlation, and increases with benchmark concentration as measured by low values of Effective *N*. In addition to identifying the relevant parameters, the derivation of the model based on benchmark and active weights helps in understanding *why* these parameters influence the amount of shorting. Of the three market parameters, the application of the model to S&P 500 data indicates that changes in security risk have historically been the most important.

When costs are considered, the analytic model also includes the assumed accuracy of the security return forecasting process as measured by the information coefficient. An increase in the ex-ante *IC* allows the manager to be more confident about the ability to offset the increased costs of shorting. Finally, the role of the number of securities in the benchmark is illustrated as we apply the analytic model to a variety of domestic and international benchmarks. A long/short strategy for a benchmark with more securities results in a larger short extension, all else equal. Even under the simplifying assumptions that allow for mathematical tractability, the basic model requires computationally intensive probability function references summed over the investable set of securities. We therefore provide a simple closed-form approximation that illustrates the intuition for the role of each parameter in the model, as well as a more robust approximation that retains its accuracy over a wide range of portfolio concentration values.

Technical Appendix

Optimal active security weights and the fundamental law

The objective in an active (as opposed to total) mean-variance portfolio optimization is to maximize the portfolio's expected active return under the budget constraint that the active weights sum to zero and that active risk (i.e., tracking error) is less than or equal to some value σ_A. The formal description of the optimization problem is

$$\text{MAX } E(R_A) = \alpha' w_A \text{ subject to } w_A' 1 = 0 \text{ and } w_A' \Omega w_A \leq \sigma_A^2 \qquad \text{(A1)}$$

where α is an Nx1 vector of forecasted security returns, w_A is an Nx1 vector of active security weights, 1 is a Nx1 vectors of ones, and Ω is an NxN security return covariance matrix. The general solution to this optimization problem gives an active weight vector of

$$w_A = \frac{\sigma_A}{\sqrt{\alpha' \Omega^{-1} \alpha}} \Omega^{-1} \alpha . \qquad \text{(A2)}$$

We employ the "full covariance matrix" version of Grinold's (1994) alpha generation process

$$\alpha = IC\, \Omega^{1/2} S \qquad \text{(A3)}$$

where IC (ex-ante information coefficient) is a scalar parameter, and S is an Nx1 vector of standard normal scores.[6] The substitution of Equation A3 into A2 gives the optimal active weight vector in terms of security scores;

$$w_A = \frac{\sigma_A}{\sqrt{N}} \Omega^{-1/2} S . \qquad \text{(A4)}$$

[6] See Clarke, de Silva, and Thorley (2006) for a discussion of full covariance matrix fundamental law mathematics, including the use of the matrix square root function. While the derivation is more complicated, we note that the results of this paper also hold for the original scalar version of Grinold's (1994) alpha generation process, $\alpha_i = IC\, \sigma_i\, S_i$, under the maintained assumption of equal pair-wise correlation coefficients for all securities.

The vector of optimal active weights in Equation A4 times the security alpha vector in Equation A3 gives an expected active portfolio return of

$$E(R_A) = IC \sqrt{N} \, \sigma_A \tag{A5}$$

which is known as the “fundamental law” of active management (Grinold and Kahn, 1994).

<u>A simplified two-parameter covariance matrix</u>

We assume a two-parameter security return covariance matrix where all the variances are equal to a single value σ^2 and all the pair-wise correlation coefficients are equal to a single value ρ. In matrix notation, this is the assertion that

$$\Omega = \sigma^2 (1-\rho)\, I + \sigma^2 \rho\, 11' \tag{A6}$$

where I is the NxN identity matrix. Under the covariance matrix assumption in Equation A6, it can be shown that the optimal active weight vector in Equation A4 reduces to the simple scalar result in Equation 2 in the body of the paper. Specifically, the budget constraint is met because the scores sum to zero, and with Equation A6 the active portfolio risk is

$$w_A' \Omega \, w_A = c^2 \sigma^2 (1-\rho)\, S'IS + c^2 \sigma^2 \rho\, S'11'S\,. \tag{A7}$$

For unit normal scores, we have the matrix products $E(S'IS) = N$ and $E(S'11'S) = 0$. We note that the expectation operator is not required in these identities if the scores are perfectly unit normal, as a group. The value of c in Equation 2 is based on these substitutions into Equation A7, and the definitional equation $w_A' \Omega \, w_A = \sigma_A^2$.

<u>Normal probability functions in the short extension model</u>

The expression in Equation 5 for the expected short weight of a single security follows from Equation 4 using well-known integral solutions in conditional probability theory, for example as used in Probit regression analysis. Using the notation Φ( ) for the standard normal cumulative density function, and φ( ) for the standard normal density function, these solutions are

$$\text{Prob}\,(S_i < -w_{B\,i}/c) \;=\; \int_{-\infty}^{-w_{B\,i}/c} \frac{1}{\sqrt{2\pi}} e^{-S^2/2} ds \;=\; \Phi\,(-w_{B\,i}/c) \tag{A8}$$

and

$$E(S_i \,|\, S_i < -w_{B\,i}/c)) \;=\; \frac{1}{\Phi(-w_{B\,i}/c)} \int_{-\infty}^{-w_{B\,i}/c} S \frac{1}{\sqrt{2\pi}} e^{-S^2/2} ds \;=\; -\frac{\varphi(-w_{B\,i}/c)}{\Phi(-w_{B\,i}/c)}\,. \tag{A9}$$

For the specific argument of $w_{Bi} = 0$, the cumulative standard normal function Φ( ) has a value of ½, and the density function φ( ) has a value of $1/\sqrt{2\pi}$. With these two values in Equation 5, the expected short weight for a zero benchmark weight security is $c/\sqrt{2\pi}$ as shown in Equation 6.

<u>Expansion equation for securities outside the benchmark</u>

As explained in Footnote 1, the basic model in Equation 7 can be used in the general case when the manager's investable universe is larger than the benchmark set. The parameter N becomes the number of securities in the larger universe, where non-benchmark securities are assigned a zero benchmark weight, with the expected short weight given in Equation 6. Alternatively, for notational precision one can disaggregate Equation 7 into benchmark and non-benchmark components. Let N_{BMK} be the number of securities in the benchmark and N_{INV} be the number of securities in the investable universe. Assuming that the benchmark is a subset of the investable universe, Equation 7 becomes

$$S_0 \;=\; \frac{c\,(N_{INV} - N_{BMK})}{\sqrt{2\pi}} \;+\; \sum_{i=1}^{N_{BMK}} c\,\varphi(-w_{B\,i}/c) - w_{B\,i}\,\Phi(-w_{B\,i}/c)\,. \tag{A10}$$

The first term of Equation A10 represents the contribution to portfolio shorting from non-benchmark securities, while the second term is the contribution from securities contained in the benchmark.

Market-neutral special case and the simple approximation

An important special case of the basic long/short model is a *market-neutral* portfolio where the benchmark weights are all zero. Using Equation 6 and substituting $w_{Bi} = 0$ for all i in Equation 7 gives Equation 8 in the body of the paper. The expected short weight of zero-benchmark-weight securities also motivates the simple approximation in Equation 9 of the paper. For a hypothetical benchmark with equal weights on the first N_E securities, and zero on the other N - N_E, the shape of the "Benchmark weights" curve in Figure 1 becomes a step function. The "Short Extension" area is a rectangle under the assumption that the benchmark weights on the first N_E securities are large enough that their expected short weights can be ignored. The height of the rectangle is the expected short weight for zero benchmark weight securities, as given in Equation 6, and the length of the rectangle is the number of zero benchmark weight securities, N – N_E. We note that ignoring the potential shorting of the first N_E securities for large N_E to N ratio benchmarks (i.e., the Russell 2000) can substantially understate expected portfolio shorting using Equation 9, and in these cases suggest using the more robust approximation in Equation 10.

Capitalization-weighted benchmark model and Effective N

To provide a simple analytic model of benchmark weights, we assume that when benchmark weights are sorted in declining magnitude, each benchmark weight is equal to the prior weight multiplied by λ

$$w_{B\,i} = \lambda \; w_{B\,i-1} = w_{B\,0} \; \lambda^{i} \,. \tag{A11}$$

The restriction that the benchmark weights sum to one, and the well-known finite geometric sum formula give a solution for w_{B0} in Equation A11, and the benchmark weight for security i is

$$w_{B\,i} = \frac{\lambda^{i-1}(1-\lambda)}{1-\lambda^{N}} \,. \tag{A12}$$

One measure of concentration used in economics is the Herfindahl Index, which in a security portfolio context is the sum of the benchmark weights squared. Effective N is the inverse of the Herfindahl Index;

$$N_E = 1 / \sum_{i=1}^{N} w_i^2 \tag{A13}$$

so that larger values of N_E indicate *less* portfolio concentration. For example, Effective N is 1 for a fully concentrated portfolio (all the weight on a single security) and N for an equally-weighted benchmark. The substitution of the analytic benchmark weights in Equation A12 into A13 gives the relationship between the geometric decline parameter, λ, and Effective N as

$$N_E = \frac{(1+\lambda)\left(1-\lambda^N\right)}{(1-\lambda)\left(1+\lambda^N\right)} \ . \tag{A14}$$

The robust approximation

The robust approximation in Equation 10 is based on the rank of an "average" security, which we designate as N_A. The absolute value of the expected active weight conditional on it being negative is

$$|\,E(c\,S\,|\,S<0))\,| = \frac{c\,\varphi(0)}{\Phi(0)} = \frac{2c}{\sqrt{2\pi}} \,. \tag{A15}$$

Setting Equation A15 equal to the benchmark weight model in Equation A12, and solving for i (with the approximation that $\lambda^N \approx 0$), gives the declining-magnitude sorted stock rank of the average security as

$$N_A \approx 1 + \frac{1}{\ln\lambda}\ln\left[\frac{\left(2c/\sqrt{2\pi}\right)}{1-\lambda}\right], \quad 0 \le N_A \le N \,. \tag{A16}$$

Defining the result in Equation A15 as $\overline{w}_A$, the probability that $\overline{w}_A$ results in a short position is

$$Prob\left(\overline{w}_A < w_B(N_A)\right) \approx 1 - \frac{N_A}{N} \,. \tag{A17}$$

Using the model benchmark weights in Equation A12 and the finite geometric sum formula, we can compute the expected benchmark weight conditional on the benchmark weight being less than $w_B(N_A)$;

$$E(w_B\,|\,w_B < w_B(N_A)) \approx \frac{1}{N-N_A}\left[1-\frac{1-\lambda^{N_A}}{1-\lambda^N}\right]. \tag{A18}$$

Using Equations A15, A17, and A18 and sorting the z-scores in ascending order, we can define shorting for the overall portfolio as the sum over the first $N/2$ (negative) z-scores

$$S_0 \approx \sum_{i=1}^{N/2} \left[\frac{1}{N-N_A} \left(1 - \frac{1-\lambda^{N_A}}{1-\lambda^N} \right) + cS_i \right] \left[1 - \frac{N_A}{N} \right]. \tag{A19}$$

Using the approximation $\lambda^N \approx 0$ and multiplying through by -1 so that that the expected portfolio shorting is a positive number, we arrive at a closed-form approximation for the expected short extension

$$S_0 \approx (N-N_A)\frac{c}{\sqrt{2\pi}} - \frac{1}{2}\lambda^{N_A} \tag{A20}$$

We wish to express A16 and A20 in terms of Effective N rather than λ. Using the approximation that $\lambda^N \approx 0$, Equation A14 can be rearranged to give

$$\lambda \approx \frac{N_E - 1}{N_E + 1}. \tag{A21}$$

Substituting Equation A21 into A16 and A20, and using the approximations $\ln(1-x) - \ln(1+x) \approx -2x$ and $\ln(1+N) \approx \ln(N)$ for large N, we express expected portfolio shorting as a function of Effective N as given in Equation 10 in the body of the paper.

Adjustment for shorting costs

While optimal active weights under costs are difficult to model, we can determine the general level of expected shorting by expanding the objective function in Equation A1. The expanded objective function adjusts the active weights in order to maximize the portfolio expected active return *after* costs. The expected active portfolio return after costs is the expected return before costs, minus the expected short extension times costs

$$\text{MAX} \;\; E(R_A) - E(S)\,(B + 2T) \tag{A22}$$

subject to the same two conditions as Equation A1; i.e., a budget constraint and a limit on the portfolio active risk. As explained in the body of the paper, B represents "borrowing" costs and T is general portfolio operating costs as determined by turnover and transaction costs. Even without an analytic solution to the active weights specified by Equation A22, we know that the expected short extension will be adjusted until the marginal value of additional shorting equals the marginal cost. In other words, a first-order equilibrium condition for the optimal solution is that the change in expected active return with respect to the level of shorting is equal to the cost of shorting,

$$\frac{\partial E(R_A)}{\partial E(S)} = B + 2T \,. \tag{A23}$$

While the exact functional relationship between expected active return and expected shorting is unknown, we know a particular point on the function, the zero-cost optimal portfolio return and shorting, and we know this point is a maximum. The zero-cost expected active portfolio return is $IC\sqrt{N}\,\sigma_A$ as shown in Equation A5, and the zero-cost expected shorting is S_0 in Equation 7. We assume a simple second-order (i.e., parabolic) functional form, which leads to a linear relationship between costs and optimal shorting. Setting the zero-cost point ($IC\sqrt{N}\,\sigma_A$, S_0) as the vertex, the general parabolic function with a maximum is

$$E(R_A) = IC\sqrt{N}\sigma_A - \frac{D}{2}\left(E(S) - S_0\right)^2 \tag{A24}$$

where D is the rate of change of the slope (i.e., the second derivative). A natural assumption that properly scales the parabola is that $D = IC\sqrt{N}\,\sigma_A / S_0$. With this substitution, we set the derivative of Equation A24 with respect to expected shorting equal to costs, as shown in Equation A23, to arrive at Equation 11 in the body of the paper.

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